DOCUMENT RESUME

ED 365 706 TM 020 892

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TITLE Type I Error Rates for Welch's Test and James's Second-Order Test under Nonnormality and Inequality of Variance when There Are Two Groups.

PUB DATE Apr 93

NOTE 22p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 12-16, 1993). Small, light type throughout document.

PUB TYPE Reports - Evaluative/Feasibility (142) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC01 Plus Postage.

DESCRIPTORS *Comparative Analysis; Equations (Mathematics); *Estimation (Mathematics); Mathematical Models; *Sample Size

IDENTIFIERS James Test; T Test; *Type I Errors; *Variance (Statistical); Welch Procedure

ABSTRACT

Type I error rates were estimated for three tests that compare means by using data from two independent samples: the independent samples t test, Welch's approximate degrees of freedom test, and James's second order test. Type I error rates were estimated for skewed distributions, equal and unequal variances, equal and unequal sample sizes, and a range of total sample sizes. Welch's test and James's test have very similar Type I error rates and tend to control the Type I error rate as well or better than the independent samples t test does. The results provide guidance about the total sample sizes required for controlling Type I error rates. Nine tables are included. (Contains 16 references.) (Author)

Type I Error Rates for Welch's Test and James's Second-Order Test Under Nonnormality and Inequality of Variance When There are Two Groups

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Paper presented at the annual meeting of

American Educational Research Association, Atlanta, April, 1993.

Abstract

Type I error rates were estimated for three tests that compare means by using data from two independent samples: the independent samples t test, Welch's approximate degrees of freedom test, and James's second-order test. Type I error rates were estimated for skewed distributions, equal and unequal variances, equal and unequal sample sizes, and a range of total sample sizes. Welch's test and James's test have very similar Type I error rates and tend to control the Type I error rate as well or better than the independent samples t test does. The results provide guidance about the total sample sizes required for controlling Type I error rates.

Type I Error Rates for Welch's Test and James's Second-Order Test Under Nonnormality and Inequality of Variance When There are Two Groups

Under variance inequality, the actual Type I error rate (τ) of the independent samples t test tends to be (a) near the nominal α level when the sample sizes are equal and sufficiently large, (b) larger than the nominal α when the smaller sample size is paired with the larger population variance (the negative condition), and (c) smaller than the nominal α level when the larger sample size is paired with the larger population variance (the positive condition) (Ramsey, 1980; Scheffe, 1959).

A number of tests have been developed to test the hypothesis H_0 $\mu_1 = \mu_2$ in a situation in which $\sigma_1^2 \neq \sigma_2^2$. Perhaps the most popular is Welch's (1938) approximate degrees of freedom, which is included in both SAS and BMDP. The test statistic for the APDF test is

$$t_v = \frac{\bar{X}_1 - \bar{X}_2}{(S_1^2/n_1 + S_2^2/n_2)^{1/2}}$$

and the critical value is a percentile of Student's t distribution with

$$f = \frac{(\sigma_1^2/n_1 + \sigma_2^2/n_2)^2}{\dfrac{\sigma_1^4}{n_1^2(n_1 - 1)} + \dfrac{\sigma_2^4}{n_2^2(n_2 - 1)}}$$

degrees of freedom. In practice, an estimate of f is obtained by replacing parameters by statistics. Welch (1951) generalized

the two-sample APDF test to K samples and Johansen (1980) generalized it to the K-sample multivariate case. In the univariate case the APDF test is typically called the Welch test.

An alternative to the APDF test is Welch's (1947) series test in which t_v is also used as a test statistic. Welch (1947) expressed the critical value for t_v as a function of S_1^2, S_2^2, and α, and developed a series solution in powers of $(n_i - 1)^{-1}$ The first three terms in the series are shown in the Table 1. The zero-order term is simply a z critical value. The first-order critical value is the sum of the zero- and first-order terms, whereas the second-order critical value is the sum of all three terms. James (1951) and James (1954) generalized the series solution to the K-sample case and the K-sample multivariate cases respectively. Consequently, tests using the series solution are referred to as James's first-order and second-order tests. Results in Wilcox (1988) and Oshima and Algina (1992) indicate that the series test has better control over τ than the APDF test does when $K > 2$.

Wilcox (1989) developed an alternative, for the univariate K-sample situation, to the Welch and James tests and reported that the test is competitive with the James second-order test in control of τ and power. However, Hsiung, Olejnik, and Huberty (in press) have shown that Wilcox's test can have excessive Type I error rates when the population means all equal but not all equal to zero.

Results reported in Harwell, Rubenstein, Hayes, and Olds (1992), Oshima and Algina (1992), Wilcox (1988, 1989), and

Wilcox, Charlin, and Thompson (1986) indicate the following:

1. With K > 2, normal distributions and inequality of variance, James's second-order test has τ closer to α than Welch's test or James's first-order test does. The advantage of the first test tends to be larger when the ratio of the largest to smallest variance is ≥ 4.

2. With symmetric non-normal distributions, τ is reasonably near α for all three tests; the performance of James's second-order test is somewhat better than that of the others.

3. With asymmetric distributions, unequal variances or unequal sample sizes or both can result in fairly large discrepancies between τ and α for all three tests . The discrepancies tend to increase as asymmetry increases and to decline as the total sample size gets larger.

The statistic t_v is a ratio of the estimated mean difference to the estimated standard error of the mean difference. As is true for all such statistics, the Type I error of t_v depends on the sampling covariance between the numerator and the square of the denominator of t_v. When this covariance is not equal to zero, extreme mean differences are associated with small standard errors and consequently the test tends to be liberal. From Kendall and Stuart (1973), the sampling covariance between the mean and variance for a single sample is μ_3/n, where μ_3 is the third central moment. Using this result, the covariance between the numerator and the square of the denominator of t_v is

$$\sigma = [\gamma_{31}\sigma_1^3/n_1^2 - \gamma_{32}\sigma_2^3/n_2^2],$$

where $\gamma_{3j} = \mu_{3j}/\sigma_j^3$. When $\gamma_{31} = \gamma_{32} = \gamma_3$ the covariance is

$$\sigma = \gamma_3[\sigma_1^3/n_1^2 - \sigma_2^3/n_2^2]. \qquad (1)$$

If the data for either group have been drawn from a distribution with a non-zero third central moment (a distribution that typically will be skewed) the sampling covariance may be non-zero and the test may be liberal. However, because the covariance goes to zero as both n_1 and n_2 go to infinity, even when γ_{31} and γ_{32} are not zero the size of the test (τ) improves asymptotically. Moreover, if $\gamma_{31} = \gamma_{32}$, the sampling covariance will be near zero and τ will be near α if σ_1 is nearly equal to σ_2 and n_1 is nearly equal to n_2.

As Harwell et al (1992) point out little is known about the magnitude of the effects of nonnormality on τ for Welch's test; the same is true for James's second-order test. Moreover, because τ for James's second-order test and for the Welch test improve with increasing sample size, information about the behavior of τ as a function of sample size will be useful in planning experiments. In this study, the behavior of τ for studies in which there are two groups and $\gamma_{31} = \gamma_{32}$ is investigated.

Method

The independent variables in the study were (a) sample-size ratio (n_2/n_1 = 3, 13/7, 1 7/13, and 1/3), (b) degree of variance inequality (σ_2/σ_1 = 3, 2, and 1), (c) distribution (beta [1.5, 8.5], exponential, and lognormal [μ = exp(.5) and σ^2 = exp(2) - exp(1)]), and (d) total sample size. The quantity γ_3 is 1.08 for the beta distribution, 2.00 for the exponential distribution, and 6.18 for the lognormal distribution. From (1), the sample size that will be required for $\hat{\tau}$ to be near α increases as γ_1 increases. Also smaller sample sizes will be required for $\hat{\tau}$ to be near α in the positive case ($n_2 > n_1$) than in the negative condition ($n_2 < n_1$). Consequently, different total sample sizes were investigated for the three distributions. The total samples sizes for the positive and negative conditions were also different. Based on a pilot study, the total samples sizes selected for the positive condition were (a) N = 20 and 40 for the beta distribution, (b) N = 20 and 40 for the exponential distribution, and (c) N = 40, 100, and 160 for the lognormal distribution. For the negative condition, the total sample sizes were (a) N = 20, 40, 60, and 80 for the beta distribution, (b) N = 20, 80, 140, and 200 for the exponential distribution, and (c) N = 100, 200, 300, 400, 500, 600, and 700 for the lognormal distribution

The data were generated by using the following steps for each replication of a condition:

1 Generate n_1 observations from the target distribution;

2 Generate n_2 observations from the target distribution;

3 Center the distributions using the population expected value.

3 Multiply each of the observations generated in step 2 by the appropriate constant to simulate heteroscedasticity.

The data were generated by using pseudorandom number generators in SAS Lognormal data were generated by using exp(z) where z was a pseudorandom normal variate generated by the RANNOR function in SAS Exponential data were generated by using

RANEXP. Suppose x and y are gamma variates, each with scale parameter one. If x and y have shape parameters a and b respectively, then x/(x + y) is a beta variate with parameters a and b. Data from the beta distribution were generated by using x/(x + y) where x and y were pseudorandom gamma variates generated by using the RANGAM function in SAS.

For each replication, the data were analyzed by using James's second-order test, the independent samples t, and Welch's test. For each of α = .01, .05, and .10, the proportion of 10000 replications that yielded significant results was recorded.

Results

In general, $\hat{\tau}$ for Welch's test and James second-order test were very similar. For example with α = .05, The largest discrepancy over all conditions in the study was .0004. Because Welch's test is simpler to apply, results for this test are reported in this paper.

In Table 2, $\hat{\tau}$ for Welch's test is reported for conditions in which the distribution was lognormal, the sample sizes were equal or resulted in the negative condition, and α = .05. Defining $d = \sigma_2/\sigma_1$ and $r = n_2/n_1$, (1) can be rewritten as

$$\sigma = \gamma_1(1 - d^2/r^2) \quad (2)$$

In the negative condition r < 1. As as result, in the negative condition and when $n_1 = n_2$, the absolute value of the covariance is an increasing function of d and γ_1 and also increases as the sample size ratio becomes more extreme. As expected from these considerations, the results in Table 2 indicate that when $d \neq 1$, $\hat{\tau} > \alpha$, with larger discrepancies occurring when d and/or n_1/n_2 is larger. The implications of the results for planning experiments depends on ones tolerance for discrepancies between τ and α. If one adopts, for example, Bradley's (1978) liberal criterion that τ should be in the interval $(.5\alpha, 1.5\alpha)$, then $N > 100$ may be required even if $n_2/n_1 = 1$. Further $N > 200$ may be required if $n_2/n_1 = 7/11$ and $N > 500$ may be required if $n_2/n_1 = 1/3$. For comparative purposes, in Table 3 $\hat{\tau}$ for t is reported. The results indicate that in comparison to use of Welch's test typically results in as good or better control of τ. The exceptions to this generalization occur when $d = 1$ and $n_2/n_1 = 1/3$. Even then, the advantage for t is not particularly marked.

Insert Tables 2 and 3 About Here

In Table 4, $\hat{\tau}$ for Welch's test is reported for conditions in which the distribution was lognormal, the sample sizes were equal or resulted in the negative condition, and $\alpha = .01$. Again the implications of the results for planning experiments depends on ones tolerance for the τ in excess of α. Adopting Bradley's (1978) liberal criterion, larger sample sizes are required for control of τ when $\alpha = .01$ in comparison to when $\alpha = .05$. This result may have implications for sample sizes when t_v is used in multiple comparisons as, for example, in Dunnett's T3. This procedure uses t_v as a test statistic and the Studentized maximum modulus critical value. Because of the large discrepancies between τ and α when $\alpha = .01$, fairly large sample sizes may be required when t_v is used for multiple pairwise comparisons with

data sampled from skewed distributions.

Insert Table 4 About Here

In Table 5, $\hat{\tau}$ for Welch's test is reported for conditions in which the distribution was exponential, the sample sizes were equal or resulted in the negative condition, and $\alpha = .05$; the results in Table 6 are for the beta distribution. The trends are similar to those for the lognormal distribution. However, because these distributions are less skewed than the lognormal distribution is, the sample size required in order to control τ is smaller. Over all conditions in Tables 5 and 6, $N = 100$ is adequate for the exponential distribution and $N = 40$ is adequate for the beta distribution.

Insert Tables 5 and 6 About Here

In Table 7, $\hat{\tau}$ for Welch's test is reported for conditions in which the sample sizes were equal or resulted in the positive condition and $\alpha = .05$. The top, middle, and lower panels contain results for the lognormal, exponential, and beta distributions, respectively. Inspection of (2) indicates that in the positive condition, the absolute value of the covariance is a j-shaped function of d and a decreasing function of r for the domains of d and r included in the study. Consistent with these trends, for a particular distribution and a fixed sample size ratio greater than one, $\hat{\tau}$ is a j-shaped function of d. For a particular distribution and value of d, $\hat{\tau}$ is a decreasing function of r. Perhaps most important, within the values of d and r studied herein, $\hat{\tau}$ is less affected by nonnormality in the positive case than it is in the negative case or even in the case of equal n. Consequently, control of τ requires a smaller total sample size in the positive case.

Insert Table 7 About Here

By fitting regression models to the $\hat{\tau}$, it may be possible to develop an equation that is useful for estimating τ for conditions not included in this study. Because τ is more disturbed in the negative and equal sample-size conditions than it is in the positive condition, regression models were fit to the $\hat{\tau}$ in Tables 2, 4 and 5. The independent variables were the $d' = \ln(d)$, $r' = \ln(r)$, $N' = \ln(N)$, and $\gamma_2' = \ln(\gamma_2)$ and products of these variables. The R^2 for the various models are reported in Table 8 and indicate that the fourth-order product is not necessary. Although R^2 for the model that included second-order products is almost equal to R^2 for the model that included second- and third-order products, the test of $N'xd'x\gamma_2'$ was significant, $t(120) = -2.09$, $p = .039$. Tests of the regression coefficients for the model with d', r', N', and γ_2', second-order products of these variables, and $N'xd'x\gamma_2'$ as independent variables indicated that only the coefficient for the $N'x\gamma_2'$ product was not significant, $t(123) = -.79$, $p = .429$. Results in Table 9 are for a model with the $N'x\gamma_2'$ product removed. The

model is consistent with both major and minor trends in the data. For fixed d', r', and γ_2' the estimated slope of the relationship between $\hat{\tau}$ and N' is

$$.005592 - d'(.015611) + r'(.011092) - d'x\gamma_2'(.001738).$$

Because $d' \geq 1$, $\gamma_2' > 1$, and $r' \leq 1$, the estimated slope is positive when $d' = r' = 1$, but becomes negative as d' increases and r' decreases. A similar analysis for γ_2' shows that the slope for γ_2' is negative for $d' = r' = 1$ but becomes positive as d' increases and r' decreases. These relationships are consistent with the results in Tables 2, 4, and 5. The slope for r' is negative for combinations of d', γ_2', and N'. The slope for d' is positive except when $\gamma_2' = 1.08$, $r = 1$ and $N' = 80$. Then it is virtually zero (-.00008). Inspection of Table 6 indicates $\hat{\tau}$ changes from .0519 to .0578 as d changes from 1 to 3.

Insert Tables 8 and 9 About Here

Summary and Conclusion

Type I error rates were estimated for the independent samples t test, Welch's APDF test, and James's second-order test. The results indicate that when the data are sampled from distributions that are skewed and the variances are different for the sampled populations, Welch's test controls the Type I error rate as well as James's second-order test controls it, at least for the range of conditions included in the study. Both tests control the Type I error rate as least as well as the independent samples t test does under these conditions. A regression equation was presented that can be used to estimate the actual Type I error rate for conditions in which the the sample sizes are equal or the sample sizes and variances have a negative relationship. Given that R^2 = .955 for the model, estimation of the Type I error rate should be fairly accurate for conditions within the range of conditions in the study. Extrapolation is less certain.

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Table 1

First Three Terms of Welch's Series Critical Value

Power of $(n_i - 1)^{-1}$	Term
Zero	z
One	$z\left[\frac{(1 + z^2)}{4} \frac{\Sigma[(S^2_i/n_i)^2/(n_i - 1)]}{[\Sigma(S^2_i/n_i)]^2}\right]^{1/2}$
Two	$z\left[-\frac{(1 + z^2)}{2} \frac{\Sigma[(S^2_i/n_i)^2/(n_i - 1)^2]}{[\Sigma(S^2_i/n_i)]^2} + \frac{(3 + 5z^2 + z^4)}{3} \frac{\Sigma[(S^2_i/n_i)^3/(n_i - 1)]^2}{[\Sigma(S^2_i/n_i)]^3} - \frac{(15 + 32z^2 + 9z^4)}{32} \frac{\{\Sigma[(S^2_i/n_i)^2/(n_i - 1)]\}^2}{[\Sigma(S^2_i/n_i)]^4}\right]^{1/2}$

Note. z denotes a fractile of the standard normal distribution.

Table 2

Estimated Type I Error Rates ($\hat{\tau}$) for Welch's Test: Lognormal Distribution and α = .05

	n_2/n_1								
	1			7/13			1/3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
100	.0407	.0638	.0807	.0525	.0897	.1009	.0633	.1030	.1166
200	.0449	.0605	.0707	.0487	.0729	.0818	.0616	.0865	.0979
300	.0488	.0561	.0657	.0528	.0687	.0697	.0626	.0746	.0862
400	.0457	.0533	.0642	.0484	.0609	.0685	.0574	.0757	.0809
500	.0462	.0551	.0600	.0534	.0604	.0642	.0579	.0694	.0755
600	.0463	.0552	.0609	.0541	.0634	.0656	.0587	.0636	.0728
700	.0518	.0509	.0570	.0528	.0569	.0608	.0547	.0685	.0680

Note. The quantity $d = \sigma_2/\sigma_1$

Table 3

Estimated Type I Error Rates ($\hat{\tau}$) for t: Lognormal Distribution and α = .05

	n_2/n_1								
	1			7/13			1/3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
100	.0416	.0649	.0817	.0430	.1157	.1636	.0453	.1569	.2316
200	.0451	.0612	.0715	.0453	.1102	.1413	.0470	.1488	.2228
300	.0488	.0563	.0662	.0485	.1088	.1347	.0475	.1579	.2178
400	.0457	.0535	.0646	.0462	.1024	.1361	.0477	.1520	.2157
500	.0462	.0544	.0602	.0506	.1043	.1333	.0522	.1566	.2071
600	.0464	.0533	.0613	.0504	.1064	.1312	.0480	.1472	.2166
700	.0519	.0510	.0572	.0484	.1009	.1291	.0479	.1542	.2058

Note. The quantity $d = \sigma_2/\sigma_1$

Table 4

Estimated Type I Error Rates $(\hat{\tau})$ for Welch's Test: Lognormal Distribution and α = .01

	n_2/n_1								
	1			7/13			1/3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
100	.0068	.0196	.0321	.0099	.0394	.0459	.0162	.0471	.0604
200	.0082	.0174	.0254	.0103	.0342	.0341	.0170	.0365	.0474
300	.0086	.0144	.0200	.0099	.0269	.0280	.0180	.0311	.0376
400	.0090	.0121	.0191	.0088	.0197	.0235	.0176	.0294	.0327
500	.0078	.0152	.0167	.0113	.0208	.0210	.0148	.0256	.0310
600	.0088	.0134	.0190	.0101	.0200	.0233	.0142	.0237	.0273
700	.0096	.0122	.0162	.0110	.0196	.0199	.0126	.0233	.0249

Note. The quantity $d = \sigma_2/\sigma_1$

Table 5

Estimated Type I Error Rates $(\hat{\tau})$ for Welch's Test: Exponential Distribution and α = .05

	n_2/n_1								
	1			7/13			1/3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
20	.0380	.0711	.0905	.0491	.0942	.0994	.0694	.1133	.1163
60	.0475	.0555	.0671	.0515	.0720	.0799	.0615	.0820	.0823
100	.0466	.0557	.0571	.0521	.0632	.0656	.0554	.0703	.0752
140	.0495	.0550	.0581	.0494	.0604	.0602	.0574	.0657	.0682

Note. The quantity $d = \sigma_2/\sigma_1$

Table 6

Estimated Type I Error Rates ($\hat{\tau}$) for Welch's Test: Beta Distribution and α = .05

	n_2/n_1								
	1			7/13			1/3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
20	.0448	.0548	.0672	.0522	.0721	.0785	.0668	.0870	.0835
40	.0491	.0531	.0581	.0527	.0622	.0664	.0571	.0663	.0692
60	.0493	.0501	.0592	.0527	.0587	.0595	.0559	.0617	.0650
80	.0519	.0519	.0578	.0500	.0525	.0595	.0565	.0601	.0599

Note. The quantity d = σ_2/σ_1

Table 7

Estimated Type I Error Rates ($\hat{\tau}$) for Welch's Test: α = .05

	n_2/n_1								
	1			13/7			3		
N	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3	d = 1	d = 2	d = 3
		Lognormal Distribution							
40	.0378	.0769	.1024	.0496	.0409	.0746	.0656	.0364	.0591
100	.0407	.0657	.0847	.0510	.0480	.0654	.0608	.0463	.0504
160	.0432	.0663	.0742	.0507	.0495	.0631	.0615	.0438	.0520
		Exponential Distribution							
20	.0380	.0716	.0905	.0491	.0455	.0650	.0684	.0354	.0465
40	.0457	.0634	.0731	.0527	.0486	.0583	.0615	.0450	.0506
		Beta Distribution							
20	.0448	.0548	.0672	.0522	.0511	.0619	.0668	.0431	.0496
40	.0491	.0531	.0581	.0527	.0475	.0514	.0571	.0483	.0547

Note. The quantity d = σ_2/σ_1

Table 8

R^2 for Several Regression Models for $\hat{\gamma}$

Model	Highest Order Product[a]	R^2
1	First	.767
2	Second	.954
3	Third	.956
4	Fourth	.957

[a] Refers to the highest-order product in the model.

Table 9

Summary of Regression Analysis for $\hat{\gamma}$

Variable	$\hat{\beta}$	$S_{\hat{\beta}}$	t	p
N′	.005592	.000904	6.18	0.0001
d′	.066313	.005081	13.05	0.0001
r′	-.049465	.003679	-13.44	0.0001
γ_3'	-.008633	.001359	-6.35	0.0001
N′xd′	-.015611	.001292	-12.07	0.0001
N′xr′	.011092	.000964	11.51	0.0001
d′xr′	-.003493	.001387	-2.51	0.0131
d′xγ_3'	.033738	.003634	9.28	0.0001
r′xγ_3'	-.015955	.001448	-11.02	0.0001
N′xd′xγ_3'	-.001738	.000723	-2.40	0.0177

Note. N′, d′, r′, and γ_3' are natural logarithms of N, d, r, and γ_3 respectively. R^2 = .955.